

Degree one cohomology with twisted coefficients of the mapping class group

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Abstract

Let Γ be the mapping class group of an oriented surface Σ of genus g with r boundary components. We prove that the first cohomology group $H^1(\Gamma, \mathcal{O}(\mathcal{M}_{\mathrm{SL}_2(\mathbb{C})})^*)$ is non-trivial, where the coefficient module is the dual of the space of algebraic functions on the $\mathrm{SL}_2(\mathbb{C})$ moduli space over Σ .

1 Introduction

Let $\Gamma = \Gamma_{g,r}$ denote the mapping class group of a compact surface $\Sigma = \Sigma_{g,r}$ with genus g and r boundary components. There is an action of Γ on the moduli space \mathcal{M}_G of flat G -connections over Σ . The vector space $\mathcal{O}(\mathcal{M}_G) \subseteq \mathrm{Fun}(\mathcal{M}_G, \mathbb{C})$ of algebraic functions on the moduli space is naturally a Γ -module. For the precise definition of the class of algebraic functions we refer to the appendix.

Presently, we consider the special case of $G = \mathrm{SL}_2(\mathbb{C})$, and we simply write \mathcal{M} for $\mathcal{M}_{\mathrm{SL}_2(\mathbb{C})}$. In this case, there is an isomorphism of Γ -modules

$$\nu: \mathcal{B}(\Sigma) \rightarrow \mathcal{O}(\mathcal{M}), \quad (1)$$

where the source denotes the algebra of BFK-diagrams on Σ : A *geometric BFK-diagram* on Σ is a finite collection of pairwise non-intersecting, non-trivial, unoriented simple loops on Σ . A BFK-diagram on Σ is an isotopy class of geometric BFK-diagrams. Letting $B = B(\Sigma)$ denote the set of BFK-diagrams on Σ , $\mathcal{B} = \mathcal{B}(\Sigma)$ is simply the complex vector space spanned by B . There is a natural algebra structure on this space; for details on this see [5] and [10]. The isomorphism ν is given on a single simple loop γ by $\nu(\gamma) = -f_{\vec{\gamma}}$, where $\vec{\gamma}$ is any of the two oriented versions of γ , and $f_{\vec{\gamma}}$ is the function which to a gauge equivalence class $[A]$ of flat connections associates the trace of the holonomy of A along $\vec{\gamma}$.

We may think of \mathcal{B} as the set of maps $B \rightarrow \mathbb{C}$ which vanish except for a finite number of diagrams. This is naturally embedded in the larger module of *all* maps $\hat{\mathcal{B}} = \text{Map}(B, \mathbb{C})$; this is clearly the same as the algebraic dual $\mathcal{O}(\mathcal{M})^*$ of $\mathcal{O}(\mathcal{M})$. The action of Γ splits B into orbits. Let S denote a set of representatives of these orbits, and for $D \in S$, let \hat{M}_D (respectively M_D) denote the space of all maps from the orbit through D to \mathbb{C} (respectively, the maps $\Gamma D \rightarrow \mathbb{C}$ which vanish for all but a finite number of diagrams in the orbit). With this notation, we obtain splittings of \mathcal{B} and $\hat{\mathcal{B}}$ as Γ -modules

$$\begin{aligned} \mathcal{O}(\mathcal{M}) &\cong \mathcal{B} \cong \bigoplus_{D \in S} M_D \\ \mathcal{O}(\mathcal{M})^* &\cong \hat{\mathcal{B}} \cong \prod_{D \in S} \hat{M}_D \end{aligned} \tag{2}$$

which induce decompositions in cohomology

$$H^*(\Gamma, \mathcal{B}) \cong \bigoplus_{D \in S} H^*(\Gamma, M_D) \tag{3}$$

$$H^*(\Gamma, \hat{\mathcal{B}}) \cong \prod_{D \in S} H^*(\Gamma, \hat{M}_D). \tag{4}$$

A cocycle $u: \Gamma \rightarrow \mathcal{O}(\mathcal{M})^* = \hat{\mathcal{B}} = \text{Map}(B, \mathbb{C})$ may also be thought as a map $u: \Gamma \times B \rightarrow \mathbb{C}$ by simply putting $u(\gamma)(E) = u(\gamma, E)$.

Theorem 1.1. *A cocycle $u: \Gamma \rightarrow \mathcal{O}(\mathcal{M})^* = \hat{\mathcal{B}} = \text{Map}(B, \mathbb{C})$ is a coboundary if and only if for each $D \in S$, the restriction of u to $\Gamma_D \times \{D\}$ is identically 0, where Γ_D denotes the stabilizer of the diagram D in Γ .*

We will use this theorem to arrive at the main result:

Theorem 1.2. *For every $g, r \geq 0$, the cohomology group $H^1(\Gamma_{g,r}, \mathcal{O}(\mathcal{M})^*)$ is a direct product of summands $H^1(\Gamma, \hat{M}_D)$, each of which is finite-dimensional. Here D runs over a set of representatives of BFK-diagrams on Σ .*

In particular, we obtain by explicit examples

Corollary 1.3. *For $g \geq 1, r \geq 0$, $H^1(\Gamma_{g,r}, \mathcal{O}(\mathcal{M})^*)$ is non-trivial.*

The motivation to study the cohomology of the mapping class group with these coefficients came from [1], particularly Proposition 6, where integrability of certain cocycles turn out to be an obstruction to finding a Γ -invariant equivalence between two equivalent star products on the moduli space. The motivation for studying that problem comes from the expectation that the star products discussed in [1] are equivalent to the star product which is constructed in [5] and which is the same as the ones induced on the $\text{SL}_2(\mathbb{C})$ -moduli space from the constructions given in [2] and [3].

This paper is organized as follows. In Section 2 we develop some of the basic properties of group cohomology which are needed in the calculations,

ending with a proof of Theorem 1.1. In Section 3, we develop an algorithm to compute $H^1(\Gamma, \hat{M}_D)$ for any BFK-diagram D , which enables us to prove Theorem 1.2. This is used in Section 4 to give a generic example of a BFK-diagram for which the cohomology is non-zero. Finally we discuss what we know when the coefficient module is $\mathcal{O}(\mathcal{M})$.

2 Group cohomological background

Theorem 2.1 (Shapiro's Lemma). *Let H be a subgroup of Γ and A a left H -module. Then there are isomorphisms*

$$H_*(H, A) \cong H_*(\Gamma, \text{Ind}_H^\Gamma A) \quad (5)$$

$$H^*(H, A) \cong H^*(\Gamma, \text{Coind}_H^\Gamma A). \quad (6)$$

Here Ind_H^Γ is the so-called *induced* module $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}H} A$, where $\mathbb{Z}\Gamma$ is considered as a right H -module via the right action of H on Γ , and the left Γ -module structure is given by $g \cdot (g' \otimes a) = gg' \otimes a$ for $g, g' \in \Gamma, a \in A$. Similarly, $\text{Coind}_H^\Gamma A$ is the co-induced module $\text{Hom}_{\mathbb{Z}H}(\mathbb{Z}\Gamma, A)$ of H -equivariant maps from the left H -module $\mathbb{Z}\Gamma$ to A . The left action of Γ is defined by

$$(g \cdot f)(g') = f(g'g)$$

for $g, g' \in \Gamma, f \in \text{Hom}_{\mathbb{Z}H}(\mathbb{Z}\Gamma, A)$.

Remark 2.2. If the action of H on A is trivial, there is a canonical bijection $\text{Hom}_{\mathbb{Z}H}(\mathbb{Z}\Gamma, A) \rightarrow \text{Map}(H \backslash \Gamma, A)$ given by $f \mapsto (Hg \mapsto f(g))$; equipping the latter with the Γ -action $(g \cdot f)(Hg') = f(Hg'g)$ this becomes an isomorphism of Γ -modules. The usual bijection between the sets of left and right cosets given by $Hg \mapsto g^{-1}H$ induces a bijection $\text{Map}(H \backslash \Gamma, A) \rightarrow \text{Map}(\Gamma/H, A)$, and the latter also carries a natural left Γ -action making this a Γ -isomorphism, namely $(g \cdot f)(g'H) = f(g^{-1}g'H)$.

We summarize the special case of Shapiro's Lemma we will need in a corollary:

Corollary 2.3. *Let A be an abelian group, and Γ a group which acts transitively on a set R . Consider the Γ -module $\text{Map}(R, A)$ of all maps $R \rightarrow A$ with action given by $(g \cdot f)(r) = f(g^{-1}r)$. Let $D \in R$ be any element, and $\Gamma_D \subseteq \Gamma$ the stabilizer subgroup of D . Then there is an isomorphism*

$$H^*(\Gamma, \text{Map}(R, A)) \cong H^*(\Gamma_D, A) \quad (7)$$

where A is considered as a trivial Γ_D -module.

Proof. The bijection $\Gamma/\Gamma_D \rightarrow R$ given by $g\Gamma_D \mapsto gD$ clearly induces an isomorphism of Γ -modules $\text{Map}(\Gamma/\Gamma_D, A) \rightarrow \text{Map}(R, A)$. Then from Shapiro's Lemma and the isomorphisms mentioned in the above remark we have a sequence of isomorphisms

$$\begin{aligned} H^*(\Gamma_D, A) &\cong H^*(\Gamma, \text{Hom}_{\mathbb{Z}\Gamma_D}(\mathbb{Z}\Gamma, A)) \\ &\cong H^*(\Gamma, \text{Map}(\Gamma_D \backslash \Gamma, A)) \\ &\cong H^*(\Gamma, \text{Map}(\Gamma/\Gamma_D, A)) \\ &\cong H^*(\Gamma, \text{Map}(R, A)). \end{aligned} \quad \square$$

Note that the Γ -module $\text{Map}(R, A)$ can also be considered as the set of all formal A -linear combinations of elements from R (that is, the sum $\sum_{r \in R} m_r r$ corresponds to the map $r \mapsto m_r$).

Specializing to the case $*$ = 1, we will now describe a more or less explicit isomorphism $H^1(\Gamma, \text{Map}(R, A)) \rightarrow H^1(\Gamma_D, A)$. First note that a map $u: \Gamma \rightarrow \text{Map}(R, A)$ can equally well be considered as a map $u: \Gamma \times R \rightarrow A$ by the adjoint formula $u(g)(r) = u(g, r)$. In this context, the cocycle condition reads

$$u(g_1 g_2, r) = u(g_1, r) + u(g_2, g_1^{-1} r). \quad (8)$$

We wish to derive necessary and sufficient conditions for a cocycle u to be a coboundary δf . For the rest of this section, fix an element $D \in R$ and let $\Gamma_D \subseteq \Gamma$ denote the stabilizer subgroup of D .

Lemma 2.4. *A cocycle $u: \Gamma \times R \rightarrow A$ is a coboundary if and only if, for every pair $g_1, g_2 \in \Gamma$ with $g_1 g_2^{-1} \in \Gamma_D$, u satisfies the condition*

$$u(g_1, D) = u(g_2, D). \quad (9)$$

Proof. First we prove the necessity of the condition. Suppose that $u = \delta f$ for some $f: R \rightarrow A$. Since the action is transitive, it is easy to see that the kernel of $\delta: C^0(\Gamma, \text{Map}(R, A)) \rightarrow C^1(\Gamma, \text{Map}(R, A))$ is the set of constant maps $R \rightarrow A$. Thus we may WLOG assume that $f(D) = 0$. Recall that $u = \delta f$ means that for every $g \in \Gamma, r \in R$ we have $u(g, r) = f(r) - f(g^{-1}r)$. In particular,

$$f(g^{-1}D) = -u(g, D) \quad (10)$$

Now if $g_1 g_2^{-1} \in \Gamma_D$, we have $g_1^{-1}D = g_2^{-1}D$, and thus $-u(g_1, D) = f(g_1^{-1}D) = f(g_2^{-1}D) = -u(g_2, D)$ as desired.

Now suppose that u satisfies (9) whenever $g_1 g_2^{-1}D = D$. We need to construct a map $f: R \rightarrow A$. For $r \in R$, choose $g \in \Gamma$ so that $g^{-1}D = r$, and define f using (10), ie. $f(r) = f(g^{-1}D) = -u(g, D)$. By assumption, this is a well-defined map (independent of the chosen g), and we only need to

check that $u = \delta f$. Let $h \in \Gamma$ and $r \in R$ be arbitrary. To calculate $(\delta f)(h, r)$, we may choose any $g \in \Gamma$ with $g^{-1}D = r$, and we obtain

$$\begin{aligned} (\delta f)(h, r) &= f(r) - f(h^{-1}r) = f(g^{-1}D) - f((gh)^{-1}D) \\ &= -u(g, D) + u(gh, D) = u(h, g^{-1}D) = u(h, r) \end{aligned}$$

by the cocycle condition (8). \square

Lemma 2.5. *The restriction of u to $\Gamma_D \times \{D\}$ is a group homomorphism $\tilde{u}: \Gamma_D \rightarrow A$.*

Proof. Let $g, h \in \Gamma_D$. Then

$$\begin{aligned} \tilde{u}(gh) &= u(gh, D) = u(g, D) + u(h, g^{-1}D) \\ &= u(g, D) + u(h, D) = \tilde{u}(g) + \tilde{u}(h) \end{aligned} \quad (11)$$

as claimed. \square

Since A is abelian, \tilde{u} factors through the abelinization $(\Gamma_D)_{\text{ab}}$ of Γ_D , and we have thus established a map $\varphi: Z^1(\Gamma, \text{Map}(R, A)) \rightarrow \text{Hom}(\Gamma_D, A) = \text{Hom}((\Gamma_D)_{\text{ab}}, A)$. The latter group may be thought of as the cohomology group $H^1((\Gamma_D)_{\text{ab}}, A)$ with trivial action of $(\Gamma_D)_{\text{ab}}$ on A .

Theorem 2.6. *The map φ factors to an isomorphism $H^1(\Gamma, \text{Map}(R, A)) \rightarrow H^1((\Gamma_D)_{\text{ab}}, A)$, which is also denoted φ .*

Before we begin the proof, we need an observation: For any cocycle u and any $g \in \Gamma$, $h \in \Gamma_D$ we have

$$\begin{aligned} u(ghg^{-1}, gD) &= u(g, gD) + u(hg^{-1}, D) \\ &= u(g, gD) + u(h, D) + u(g^{-1}, D) \\ &= u(h, D) \end{aligned}$$

using $h^{-1}D = D$ and the fact that $0 = u(1) = u(g^{-1} \cdot g) = u(g^{-1}) + g^{-1}.u(g)$.

Proof (of Theorem 2.6). To prove the first part of the theorem, we need to show that the restriction of a coboundary δf to $\Gamma_D \times \{D\}$ is identically 0. But this is trivial since

$$\widetilde{\delta f}(h) = (\delta f)(h, D) = f(D) - f(h^{-1}D) = 0$$

for $h \in \Gamma_D$.

Next, assume that the cocycle u restricts to the zero homomorphism $\Gamma_D \rightarrow A$. Then for any two elements $g_1, g_2 \in \Gamma$ with $g_1 g_2^{-1} \in \Gamma_D$ we have

$$\begin{aligned}
 0 &= u(g_1 g_2^{-1}, D) \\
 &= u(g_1, D) + u(g_2^{-1}, g_1^{-1} D) \\
 &= u(g_1, D) + u(g_2^{-1})(g_1^{-1} D) \\
 &= u(g_1, D) - g_2^{-1} \cdot u(g_2)(g_1^{-1} D) \\
 &= u(g_1, D) - u(g_2)(g_2 g_1^{-1} D) \\
 &= u(g_1, D) - u(g_2, D)
 \end{aligned}$$

since $g_2 g_1^{-1} = (g_1 g_2^{-1})^{-1} \in \Gamma_D$, and by Lemma 2.4 we see that u is a coboundary. This shows that φ is injective.

Now, for surjectivity, let $u: \Gamma_D \rightarrow A$ be any homomorphism. We need to extend u to all of $\Gamma \times R$ in such a way that it becomes a cocycle. To produce this extension, we first assume that an extension exists, and use this to write a formula for a cocycle cohomologous to the given extension. Then we prove that this formula actually defines a cocycle.

Choose a collection $\{h_i\}_{i \in I}$ of representatives for the set $\Gamma_D \backslash \Gamma$ of *right* cosets of Γ_D , and let $1 \in \Gamma$ represent the coset Γ_D . Recall that the map $\Gamma_D \backslash \Gamma \rightarrow \Gamma / \Gamma_D$ given by $\Gamma_D x \mapsto x^{-1} \Gamma_D$ is a bijection between the set of right cosets and the set of left cosets of Γ_D . In particular, $\{h_i^{-1}\}_{i \in I}$ is a collection of representatives of the set of left cosets. We also have a bijection $\Gamma / \Gamma_D \rightarrow R$ given by $x \Gamma_D \mapsto xD$. Now, for any coboundary δf with which we alter u , we may (as has been used a couple of times) WLOG assume that $f(D) = 0$. Then the formula $(\delta f)(h_i)(D) = f(D) - f(h_i^{-1} D) = -f(h_i^{-1} D)$ and the fact that $i \mapsto h_i^{-1} D$ is a bijection $I \rightarrow R$ show that we may assume that the extension u satisfies $u(h_i, D) = 0$ for $i \in I$. Furthermore, u is uniquely determined by its cohomology class and this requirement.

The cocycle condition implies that

$$u(gh_i, D) = u(g, D) + u(h_i, g^{-1} D) = u(g, D) \quad (12)$$

for $i \in I$ and $g \in \Gamma_D$. Since every $x \in \Gamma$ admits a unique factorization as $x = gh_i$ for some $i \in I$ and $g \in \Gamma_D$, this formula extends u to all of $\Gamma \times \{D\}$.

Now consider any $x \in \Gamma$ and $E \in R$. There is a unique $j \in I$ with $h_j^{-1} D = E$, and we have $\Gamma_E = h_j^{-1} \Gamma_D h_j$. Furthermore, the collection $\{h_j^{-1} h_i h_j\}_{i \in I}$ is a collection of representatives for the set $\Gamma_E \backslash \Gamma$ of right cosets of Γ_E . This means that we may factorize x uniquely as $(h_j^{-1} g_0 h_j)(h_j^{-1} h_i h_j)$ for some $g_0 \in \Gamma_D$ and $i \in I$. Now we calculate

$$u(x, E) = u(h_j^{-1} g_0 h_j \cdot h_j^{-1} h_i h_j, h_j^{-1} D) \quad (13)$$

$$= u(h_j^{-1} g_0 h_j, h_j^{-1} D) + u(h_j^{-1} h_i h_j, h_j^{-1} g_0^{-1} h_j h_j^{-1} D) \quad (14)$$

By the observation preceding this proof (with $g = h_j^{-1}$ and $h = g_0$), the first term is equal to the known quantity $u(g_0, D)$. For the second term, we apply the cocycle condition a few more times:

$$\begin{aligned} u(h_j^{-1}h_ih_j, h_j^{-1}D) &= u(h_j^{-1}, h_j^{-1}D) + u(h_ih_j, D) \\ &= -u(h_j, D) + u(h_ih_j, D) \\ &= u(h_ih_j, D) \end{aligned}$$

which is also known since u is known on $\Gamma \times \{D\}$. Thus our formula for the extension of u to all of $\Gamma \times R$ reads

$$u(x, E) = u(g_0, D) + u(h_ih_j, D) \quad (15)$$

where $j \in I$ is the unique index such that $h_j^{-1}D = E$, $i \in I$ is the unique index so that x belongs to the right coset of Γ_E represented by $h_j^{-1}h_ih_j$, and $g_0 = h_jgh_j^{-1}$ is the unique element in Γ_D such that $x = g(h_j^{-1}h_ih_j) = (h_j^{-1}g_0h_j)(h_j^{-1}h_ih_j)$. The second term above is defined by (12); thus one must find the $k \in I$ such that h_ih_j is an element of the right coset of Γ_D represented by h_k , say $h_ih_j = g_1h_k$ for $g_1 \in \Gamma_D$, and then $u(h_ih_j, D) = u(g_1, D)$. It remains to check that (15) defines a cocycle.

Let $x, y \in \Gamma$ and $E \in R$ be arbitrary. As above, there is a unique $j \in I$ with $h_j^{-1}D = E$. Let's try to calculate the right-hand side of the cocycle condition $u(xy, E) = u(x, E) + u(y, x^{-1}E)$. We must choose $i \in I$ and $g_1 \in \Gamma_D$ such that

$$x = (h_j^{-1}g_1h_j)(h_j^{-1}h_ih_j) \quad (16)$$

and next we choose $k \in I$ and $g_2 \in \Gamma_D$ such that $h_ih_j = g_2h_k$. Then

$$u(x, E) = u(g_1, D) + u(g_2, D) = u(g_1g_2, D)$$

Now, the element $x^{-1}E$ of R is the same as

$$x^{-1}E = h_j^{-1}h_i^{-1}g_1^{-1}h_jE = h_j^{-1}h_i^{-1}D = (h_ih_j)^{-1}D = (g_2h_k)^{-1}D = h_k^{-1}D$$

so in the calculation of $u(y, x^{-1}E)$ it is h_k which plays the role as h_j in the recipe. This recipe then requires us to find $g_3 \in \Gamma_D$ and $\ell \in I$ such that

$$y = (h_k^{-1}g_3h_k)(h_k^{-1}h_\ell h_k), \quad (17)$$

and $g_4 \in \Gamma_D$ and $m \in I$ such that $h_\ell h_k = g_4h_m$. Then

$$u(y, x^{-1}E) = u(g_3, D) + u(g_4, D) = u(g_3g_4, D).$$

Multiplying x and y using the expressions (16) and (17) and the relations defining the various h 'es we obtain

$$\begin{aligned} xy &= (h_j^{-1}g_1h_ih_j)(h_k^{-1}g_3h_\ell h_k) \\ &= h_j^{-1}g_1g_2g_3g_4h_m \end{aligned} \quad (18)$$

On the other hand, the recipe requires us to choose $g \in \Gamma_D$ and $n \in I$ such that

$$xy = h_j^{-1}gh_jh_j^{-1}h_nh_j, \quad (19)$$

and $g' \in \Gamma_D$ and $p \in I$ such that $h_nh_j = g'h_p$. Then $u(xy, E) = u(g, D) + u(g', D)$. Comparing (18) and (19) we see that $g_1g_2g_3g_4h_m = gh_nh_j$, showing that (by uniqueness of g' and p) $h_p = h_m$ and

$$g' = g^{-1}g_1g_2g_3g_4 \quad (20)$$

Finally we conclude that

$$\begin{aligned} u(xy, E) &= u(g, D) + u(g', D) \\ &= u(g_1g_2g_3g_4, D) \\ &= u(g_1g_2, D) + u(g_3g_4, D) \\ &= u(x, E) + u(y, x^{-1}E) \end{aligned}$$

showing that the given recipe in fact defines a cocycle $u: \Gamma \times R \rightarrow A$. The proof is complete. \square

Proof (of Theorem 1.1). By the splitting (4), a cocycle $u: \Gamma \rightarrow \hat{B}$ is the same as a collection of cocycles $u_D: \Gamma \rightarrow \hat{M}_D$ for $D \in S$. In fact, thinking of u as a map $\Gamma \times B \rightarrow \mathbb{C}$, u_D is simply the restriction of u to $\Gamma \times (\Gamma D)$. Specializing Theorem 2.6 to the case $A = \mathbb{C}$ and $R = \Gamma D$, we see that each u_D is a coboundary if and only if u_D restricted to $\Gamma_D \times \{D\}$ is zero. \square

In section 3 below, we are going to need a theorem linking the low-dimensional homology groups of the groups appearing in a short exact sequence. Again quoting from [4] (Corollary VII.6.4)

Theorem 2.7. *Let $1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$ be a short exact sequence of groups, and M a B -module. Then there is an exact sequence of low-dimensional homology groups*

$$\begin{aligned} H_2(B, M) &\rightarrow H_2(C, M_A) \rightarrow \\ H_1(A, M)_C &\rightarrow H_1(B, M) \rightarrow H_1(C, M_A) \rightarrow 0. \end{aligned} \quad (21)$$

Here we regard M as an A -module via restriction of scalars, and then clearly $C \cong B/A$ acts on the co-invariants group M_A , making sense of $H_*(C, M_A)$. Since A is a normal in B , conjugation by $b \in B$ defines an action on A by automorphisms, so there is an induced action on homology $c(b)_*: H_*(A, M) \rightarrow H_*(A, M)$. One may show that A acts trivially, so there is an induced action of C , and we have $H_1(A, M)_B = H_1(A, M)_C$.

3 Computing $H^1(\Gamma, \mathcal{O}(\mathcal{M})^*)$

By the Γ -equivariant isomorphism (1) and the splitting (4), it is clear that $H^1(\Gamma, \mathcal{O}(\mathcal{M})^*)$ splits as a direct product of $H^1(\Gamma, \hat{M}_D)$ -s, proving the first part of Theorem 1.2. In order to prove that these are all finite-dimensional, we develop in this section an algorithm to compute them. Then it suffices to find a single diagram D for which $H^1(\Gamma, \hat{M}_D)$ is non-zero in order to prove Corollary 1.3.

Recall that $\hat{M}_D = \text{Map}(\Gamma D, \mathbb{C})$. By the previous section (specifically Corollary 2.3), we have $H^1(\Gamma, \hat{M}_D) \cong H^1(\Gamma_D, \mathbb{C}) = \text{Hom}(\Gamma_D, \mathbb{C})$, so since \mathbb{C} is abelian and torsion-free, to compute $H^1(\Gamma, \hat{M}_D)$ amounts to computing the first homology group of the stabilizer Γ_D with rational coefficients.

In order to compute $H_1(\Gamma_D, \mathbb{Q})$, we consider the surface Σ' which is obtained from Σ by cutting along D . Let n denote the number of components of D , and let n' be the maximal number of components of D such that Σ cut along these is still connected. Put $n = n' + n''$. Then Σ' is a (possibly non-connected) surface with $1 + n''$ connected components, total genus $g' = g - n'$ and a total of $r' = r + 2n$ boundary components. There is a “glueing map” $j: \Sigma' \rightarrow \Sigma$ which is a local diffeomorphism away from the $2n$ boundary components arising from D . The mapping class group Γ' of Σ' maps to Γ_D (via j), because for any homeomorphism $\gamma': \Sigma' \rightarrow \Sigma'$ fixed on the boundary $\partial\Sigma'$, there is a unique homeomorphism $\gamma: \Sigma \rightarrow \Sigma$ fixing $\partial\Sigma$ and fitting into the diagram

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\gamma'} & \Sigma' \\ \downarrow j & & \downarrow j \\ \Sigma & \xrightarrow{\gamma} & \Sigma \end{array} \quad (22)$$

and clearly any isotopy fixed on $\partial\Sigma'$ descends to an isotopy fixed on $\partial\Sigma$. This group homomorphism is not injective, since a diffeomorphism of Σ' consisting of “oppositely oriented” Dehn twists along two boundary components glued together by j is isotopic to the identity in $\text{Diff}(\Sigma)$. Also, it is not surjective, since the elements of Γ_D are allowed to permute the components of D , which no homeomorphism coming from Γ' can do.

Hence, we need a notion of a “larger” mapping class group through which information about $H_*(\Gamma')$ can be translated into information about

$H_*(\Gamma_D)$. To this end, choose an oriented parametrization $c: \bigsqcup_{2n+r} S^1 \rightarrow \partial\Sigma'$ of the boundary. Consider the group $\text{Diff}(\Sigma'; c)$ of diffeomorphisms of Σ' preserving this parametrization, ie. the group of diffeomorphisms γ such that $c^{-1} \circ \gamma|_{\partial\Sigma'} \circ c$ is a permutation of the $2n+r$ copies of S^1 consisting of identity maps. We define G' to be the group $\text{Diff}(\Sigma'; c)$ modulo isotopies preserving c . Note that in case Σ' is non-connected, elements of G' are allowed to permute homeomorphic components.

There is a homomorphism from G' to the permutation group of the set of boundary components of Σ' , and the kernel of this map is easily seen to be Γ' (if a diffeomorphism maps each boundary component to itself and at the same time preserves a parametrization, it fixes the boundary point-wise). Thus we have a short exact sequence

$$1 \longrightarrow \Gamma' \longrightarrow G' \longrightarrow P' \longrightarrow 1,$$

where P' is the appropriate subgroup of $S_{\pi_0\partial\Sigma'}$. (In case Σ' is connected, any permutation of the boundary components is realizable through a diffeomorphism.)

Consider the subgroup $Q' \subseteq P'$ of permutations which fix $\pi_0(\partial\Sigma)$ (or more precisely the set $\pi_0(j^{-1}(\partial\Sigma))$) and preserves the pairing of elements of $\pi_0(\partial\Sigma' - j^{-1}(\partial\Sigma))$ induced by j . In other words, Q' consists of the permutations $\sigma \in P'$ such that $\sigma(b) \in \pi_0(j^{-1}(j(b)))$ for every boundary component b of Σ' ; ie. $\sigma(b) = b$ if b is a boundary component of Σ , otherwise $\sigma(b)$ is either equal to b or the boundary component of Σ' which it is identified with by j . Let H' be the pre-image of Q' in G' , so we have a new exact sequence

$$1 \longrightarrow \Gamma' \longrightarrow H' \longrightarrow Q' \longrightarrow 1. \quad (23)$$

One could also define H' as the subgroup of G' consisting of elements which descend to elements of Γ_D as in (22) above. Now, the homomorphism $H' \rightarrow \Gamma_D$ is easily seen to be surjective. For the moment, assume that the kernel of this map is the free abelian group \mathbb{Z}^n with one generator for each component of D . Then we have another short exact sequence

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow H' \longrightarrow \Gamma_D \longrightarrow 1. \quad (24)$$

Lets apply Theorem 2.7 to (23). As explained earlier, we use rational coefficients (with trivial action), so since Q' is a finite group, its rational homology (in positive dimensions) vanishes, and we are left with

$$0 \longrightarrow H_1(\Gamma'; \mathbb{Q})_{Q'} \longrightarrow H_1(H'; \mathbb{Q}) \longrightarrow 0 \quad (25)$$

By work of (among others) Harer, the low-dimensional homology groups of mapping class groups are known, at least with rational coefficients. For easy reference, we collect the results we will need in a proposition.

Proposition 3.1. *Let $\Gamma_{g,r}$ denote the mapping class group of a genus g surface with r boundary components.*

- (1) *If $g \geq 2$, $H_1(\Gamma_{g,r}; \mathbb{Q}) = 0$.*
- (2) *For $g = 1$, $H_1(\Gamma_{1,r}; \mathbb{Q}) \cong \mathbb{Q}^r$.*
- (3) *For $g = 0$, $H_1(\Gamma_{0,r}; \mathbb{Q}) \cong \mathbb{Q}^{(r-1)r/2}$.*

In fact, (1) is easy to prove using simple geometric considerations, and the fact that the mapping class group is generated by Dehn twists (cf. the appendix, Corollary A.8), except that in genus 2 one has to rely on a presentation of the mapping class group. For proofs of (2) and (3) we refer to [7]. Harer's work even give explicit generators: In the case (2), the Dehn twists along the r boundary components represent a basis for the rational homology. In the case (3), think of $\Sigma_{0,r}$ as the closed unit disc with $r - 1$ small open discs centered at the x -axis removed. Then the $r - 1$ Dehn twists along the boundaries of these small discs, along with $\binom{r-1}{2} = (r - 2)(r - 1)/2$ twists along circles enclosing exactly two of these discs represent a basis for $H_1(\Gamma_{0,r}; \mathbb{Q})$.

Since the mapping class group of a non-connected surface, where each connected component has at least one boundary component, is obviously the product of the mapping class groups of the components, this shows how to find a set of generators for $H_1(\Gamma'; \mathbb{Q})$, and that we may in fact represent these generators by Dehn twists. Since the action of Q' is induced by the conjugation action of H' on Γ' , we see that the action simply identifies some of these generators. The exact details regarding which generators are thus identified depend on topological constraints (for instance, it can happen that two components of Σ' are homeomorphic, but that there does not exist a glueing-compatible diffeomorphism taking one to the other).

Applying Theorem 2.7 to (24) and using the surjective map $H_1(A, M) \rightarrow H_1(A, M)_C$, we obtain another exact sequence

$$H_1(\mathbb{Z}^n; \mathbb{Q}) \longrightarrow H_1(H'; \mathbb{Q}) \longrightarrow H_1(\Gamma_D; \mathbb{Q}) \longrightarrow 0,$$

which by using the isomorphism (25) becomes

$$H_1(\mathbb{Z}^n; \mathbb{Q}) \longrightarrow H_1(\Gamma'; \mathbb{Q})_{Q'} \longrightarrow H_1(\Gamma_D; \mathbb{Q}) \longrightarrow 0. \quad (26)$$

Recall that the group \mathbb{Z}^n really means the free abelian group generated by n pairs of left and right Dehn twists in the boundary components arising from the cutting along D . Consider such a pair of Dehn twists $\tau_b, \tau_{b'}^{-1}$ in the boundary curves b, b' which are glued together by j . If both b and b' belong to components of Σ' with genus ≥ 2 , $\tau_b \tau_{b'}^{-1}$ is mapped to 0 in $H_1(\Gamma'; \mathbb{Q})_{Q'}$. If exactly one of them belongs to a component with genus ≥ 2 , the product $\tau_b \tau_{b'}^{-1}$ is mapped to a generator of $H_1(\Gamma'; \mathbb{Q})_{Q'}$, and if both b, b' belong to

genus ≤ 1 components, the image of $\tau_b \tau_{b'}^{-1}$ in $H_1(\Gamma'; \mathbb{Q})_{Q'}$ identifies the generators $\tau_b, \tau_{b'}$ (in case these were not already identified by the action of Q').

In this way we see that we have a combinatorial method to compute $H_1(H; \mathbb{Q})$ by cutting along D , writing down all generators coming from genus 0 and 1 components, and identifying and/or removing generators according to which permutations of the generators are topologically realizable or which are killed by the image of $H_1(\mathbb{Z}^n; \mathbb{Q})$.

Proposition 3.2. *For a BFK-diagram D on Σ , $H^1(\Gamma, \hat{M}_D) \cong \mathbb{C}^n$, where n is the dimension of the rational vector space $H_1(\Gamma_D; \mathbb{Q})$. It is always finite, and may be found by the algorithm described above.*

This in particular proves the second claim in Theorem 1.2.

4 An example

We now wish to construct a diagram D such that the factor $H^1(\Gamma, \hat{M}_D)$ in (4) is non-trivial, ie. such that there exists a non-zero homomorphism $\Gamma_D \rightarrow \mathbb{C}$. In view of (26) and Proposition 3.1 above, we must choose D such that Σ' contains at least one component of genus at most 1. With this in mind, we arrive at the following generic example.

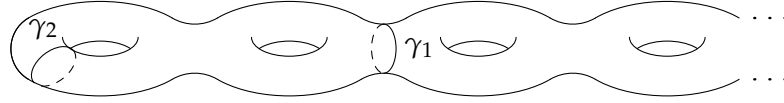


Figure 1: A two-component BFK-diagram.

Consider a surface $\Sigma_{g,r}$ of genus $g \geq 4$ with an arbitrary number, r , of boundary components. We may choose a two-component BFK-diagram $D = \gamma_1 \cup \gamma_2$ such that γ_1 is a separating curve, dividing Σ into a surface of genus 2 with one boundary component (arising from the cut), and a surface of genus $g - 2$ containing all the original boundary components. The curve γ_2 is chosen such that it is a non-separating (hence genus-decreasing) curve in the genus 2 component. Hence in this case, the cut surface Σ' consists of a genus 1 component Σ'_1 with three boundary components (one arising from the cut along γ_1 , the other two being the boundaries arising from the cut along γ_2) and a genus $g - 2$ component Σ'_2 with $r + 1$ boundary components. We denote the boundary curves arising from the cut along γ_1 by η_{11} and η_{12} , respectively (with η_{11} belonging to Σ'_1), and the boundaries arising from the cut along γ_2 are similarly denoted η_{21}, η_{22} .

Now clearly the mapping class group Γ' is the product $\Gamma_{g-2, r+1} \times \Gamma_{1,3}$, and since $g \geq 4$, the rational homology of the first factor vanishes. By

Proposition 3.1, $H_1(\Gamma'; \mathbb{Q}) = H_1(\Gamma_{1,3}; \mathbb{Q}) = \mathbb{Q}\{\eta_{11}, \eta_{21}, \eta_{22}\}$, the rational vector space spanned by the Dehn twists $\eta_{11}, \eta_{21}, \eta_{22}$ (in order to keep notation simple we denote a curve and the Dehn twist along it by the same symbol). It is clear that the only (non-trivial) permutation of the boundary components of Σ' that is compatible with glueing is the interchange of η_{21} and η_{22} , and the effect of $Q' \cong \mathbb{Z}/2$ on $H_1(\Gamma'; \mathbb{Q})$ is hence to identify the two generators η_{21}, η_{22} , so we may think of $H_1(\Gamma'; \mathbb{Q})_{Q'}$ as the rational vector space spanned by η_{11} and η_{21} .

The two generators of $H_1(\mathbb{Z}^2; \mathbb{Q})$ may be chosen to be $\eta_{11} - \eta_{12}$ and $\eta_{21} - \eta_{22}$. We have just established that $\eta_{21} = \eta_{22}$ in $H_1(\Gamma'; \mathbb{Q})_{Q'}$, so the latter of these generators is mapped to 0. We also have that $\eta_{12} = 0$ in $H_1(\Gamma'; \mathbb{Q})_{Q'}$, so the first generator is simply mapped to η_{11} . Hence by the exactness of (26), $H_1(\Gamma_D; \mathbb{Q})$ is a 1-dimensional rational vector space spanned by the Dehn twist in γ_2 , so for this particular diagram D , we see that $H^1(\Gamma, \hat{M}_D) \cong \mathbb{C}$.

It is not hard to obtain similar examples for the remaining low values of genus. If $g = 3$, we may choose $D = \gamma_1 \cup \gamma_2$ as above, dividing the surface into two genus 1 components, one with three boundary components arising from the cut along D and one with the original boundary components (if any) together with the one arising from the cut along γ_1 . The only difference is that the other component now also contributes to the homology; it is still true that γ_2 survives to represent a non-zero element of $H_1(\Gamma_D; \mathbb{Q})$.

When $g = 2$, we may simply choose D to consist of a single non-separating curve γ . Then Σ' is a connected genus 1 surface with $r + 2$ boundary components. The rational homology of Γ' is thus \mathbb{Q}^{r+2} , and we see that $H_1(\Gamma_D, \mathbb{Q})$ has dimension $r + 1$, spanned by the Dehn twists in the boundary components and γ .

In the remaining case of $g = 1$, first assume $r \geq 1$, and let D consist of a single curve parallel to a boundary component. Then obviously $\Gamma_D = \Gamma$, and $H_1(\Gamma_D, \mathbb{Q}) = H_1(\Gamma, \mathbb{Q}) = \mathbb{Q}^r$. Finally, in the special case of a closed torus, we refer to the example given in the next section of a cocycle with values in the module $\mathcal{O}(\mathcal{M})$.

5 Algebraic coefficients

Although the results in the present paper indicate that $H^1(\Gamma, \mathcal{O}(\mathcal{M})^*)$ is not trivial, this does not necessarily imply that the same holds true for the cohomology $H^1(\Gamma, \mathcal{O}(\mathcal{M}))$ with algebraic functions as coefficients.

In the simple case of a closed torus, there is an example of a cocycle with values in $\mathcal{O}(\mathcal{M})$ which cannot be a coboundary. Namely, consider the well-known presentation of $\Gamma_{1,0} \cong \mathrm{SL}_2(\mathbb{Z})$ as $\langle \tau_\alpha, \tau_\beta \mid \tau_\alpha \tau_\beta \tau_\alpha = \tau_\beta \tau_\alpha \tau_\beta, (\tau_\alpha \tau_\beta)^6 = 1 \rangle$. The elements τ_α and τ_β may be realized as Dehn twists in curves α, β intersecting transversely in a single point. We now define a cocycle u on the generators by $u(\tau_\alpha) = \alpha - \beta$ and $u(\tau_\beta) = \beta - \alpha$, where on the right hand

sides we consider α and β as 1-component BFK-diagrams. It is easy to check that this in fact defines a cocycle, since

$$\begin{aligned} u(\tau_\alpha \tau_\beta) &= (\alpha - \beta) + \tau_\alpha(\beta - \alpha) = -\beta + \tau_\alpha \beta \\ u(\tau_\beta \tau_\alpha) &= (\beta - \alpha) + \tau_\beta(\alpha - \beta) = -\alpha + \tau_\beta \alpha \end{aligned}$$

so

$$\begin{aligned} u(\tau_\beta \tau_\alpha \tau_\beta) &= (\beta - \alpha) + \tau_\beta(-\beta + \tau_\alpha \beta) = -\alpha + \tau_\beta \tau_\alpha \beta \\ u(\tau_\alpha \tau_\beta \tau_\alpha) &= (\alpha - \beta) + \tau_\alpha(-\alpha + \tau_\beta \alpha) = -\beta + \tau_\alpha \tau_\beta \alpha \end{aligned}$$

But by Lemma A.3, we have $\tau_\alpha \tau_\beta \alpha = \beta$ and $\tau_\beta \tau_\alpha \beta = \alpha$, so both right hand sides are 0, and u satisfies the first relation. Now it is trivial to see that it also satisfies the second, because by the first relation it may also be written $(\tau_\alpha \tau_\beta)^6 = (\tau_\alpha \tau_\beta \tau_\alpha)^4 = 1$, so we have

$$u(\psi^4) = u(\psi) + \psi u(\psi) + \psi^2 u(\psi) + \psi^3 u(\psi) = 0,$$

where $\psi = \tau_\alpha \tau_\beta \tau_\alpha$.

It is clear that u is not a coboundary, because for every linear combination f of BFK-diagrams, the coefficient of α in $(\delta f)(\tau_\alpha) = f - \tau_\alpha f$ is necessarily 0. This proves that $H^1(\Gamma_{1,0}, \mathcal{O}(\mathcal{M})) \neq 0$. It is interesting to see if this example can be generalized, for example using the simple presentation of $\Gamma_{g,r}$ given in [6].

A Appendix

A.1 The moduli space

Let $P_i, i \in I$, be a collection of pair-wise non-isomorphic principal G -bundles over Σ , such that any principal G -bundle is isomorphic to some (clearly unique) P_i . We let $\mathcal{A}_{P_i}^F \subset \mathcal{A}_{P_i}$ denote the space of flat connections. The gauge group $\mathcal{G}_{P_i} = \text{Aut}(P_i)$ acts on this space, and we let $\mathcal{M}_{P_i} = \mathcal{A}_{P_i}^F / \mathcal{G}_{P_i}$. We then define the moduli space of flat G -connections over Σ to be $\mathcal{M} = \bigsqcup_{i \in I} \mathcal{M}_{P_i}$.

Choosing a basepoint $x \in \Sigma$, the *representation variety* is the space

$$\mathcal{R} = \text{Hom}(\pi_1(\Sigma, x), G) / G, \quad (27)$$

where the action of G is by post-conjugation. It is well-known that there is a bijection $R: \mathcal{M} \rightarrow \mathcal{R}$ given as follows: For each $i \in I$, choose some p_i in the fibre of P_i over x . For a gauge equivalence class $[A]$ of flat connections in P_i and a homotopy class $[\gamma]$ of loops based at x , $R([A])([\gamma]) \in G$ is the holonomy along γ with respect to A . This defines a homomorphism $R([A]): \pi_1(\Sigma, x) \rightarrow G$, and the dependence on the choice of points p_i vanish when we pass to the quotient $\text{Hom}(\pi_1(\Sigma, x), G) / G$.

If we choose a finite presentation $\langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$ of $\pi_1(\Sigma, x)$, we may identify $\text{Hom}(\pi_1(\Sigma, x), G)$ with a certain closed subset H of $G^{\times n}$, namely the n -tuples (A_1, \dots, A_n) such that $r_i(A_1, \dots, A_n) = 1$ for all i . If G is an algebraic group, we may consider the ring \mathcal{O} of algebraic functions on $G^{\times n}$, and inside this the ideal $I(H)$ of functions vanishing on H . The ring of algebraic functions on H is the quotient ring $\mathcal{O}/I(H)$. Since the representation variety is identified with H/G , we define the algebraic functions on the representation variety to be the set $\mathcal{O}^G/I(H)$; the space of algebraic functions on G^n which are invariant under conjugation, modulo the ideal vanishing on H . This is, in fact, independent of the chosen presentation of π_1 , whence there is a well-defined notion of algebraic functions on the moduli space.

A.2 The mapping class group

The mapping class group Γ of an oriented surface Σ may be defined as the group $\text{Diff}(\Sigma, \partial)$ of orientation-preserving diffeomorphisms of Σ fixing the boundary pointwise, modulo isotopies fixing the boundary. Since any homeomorphism is isotopic to a diffeomorphism, and two diffeomorphisms are isotopic through diffeomorphisms if and only if they are isotopic through homeomorphisms, we may also unambiguously speak of the isotopy class of a homeomorphism of Σ .

It is well-known that Γ is generated by the isotopy classes of a certain “twist diffeomorphisms” known as Dehn twists, which may be defined as follows: Let A be the annulus given in polar coordinates (r, θ) by $1 \leq r \leq 2$, and choose some smooth, increasing function $\varphi: [1, 2] \rightarrow [0, 2\pi]$. The standard twist diffeomorphism of A is given in polar coordinates by $(r, \theta) \mapsto (r, \theta + \varphi(r))$. This diffeomorphism fixes ∂A point-wise, and its isotopy class is independent of the choice of φ . Now, if α is some simple closed curve on Σ , we may choose an orientation-preserving embedding of A in Σ such that the inner boundary component coincides with α . Then a twist along α is obtained by extending the standard twist of A by the identity to the rest of Σ . (A priori, this may only define a homeomorphism, but one could for instance require φ to be constant near 1 and 2). The isotopy class of this twist is independent of the embedding, and depends only on the isotopy class of α . It is known as the Dehn twist τ_α along α . We list a few easy facts about Dehn twists, the first of which is geometrically obvious. For proof we refer to section 4 of [8].

Lemma A.1. *Dehn twists on non-intersecting curves commute.* □

Lemma A.2. *If h is a diffeomorphism of Σ , the relation $h\tau_\gamma h^{-1} = \tau_{h\gamma}$ holds in Γ .*

Lemma A.3. If α and β are (isotopy classes of) simple closed curves intersecting transversely in a single point, then $\tau_\alpha \tau_\beta \alpha = \beta$, and the Dehn twists are braided, ie. satisfy $\tau_\alpha \tau_\beta \tau_\alpha = \tau_\beta \tau_\alpha \tau_\beta$.

Lemma A.4. Consider the surface $\Sigma_{0,4}$, ie. a sphere with four holes. Let γ_i denote the i 'th boundary component, $0 \leq i \leq 3$, and γ_{ij} a loop enclosing the i 'th and j 'th boundary components, $1 \leq i < j \leq 3$. Let $\tau_i = \tau_{\gamma_i}$ and $\tau_{ij} = \tau_{\gamma_{ij}}$. Then

$$\tau_0 \tau_1 \tau_2 \tau_3 = \tau_{12} \tau_{13} \tau_{23}. \quad (28)$$

This is known as the *lantern relation*.

Proof. We may also regard $\Sigma_{0,4}$ as a disc with three open discs removed as in Figure 2a. Connect γ_0 with γ_i by a small arc I_i , $i = 1, 2, 3$, such that the three arcs are disjoint. Then $\Sigma_{0,4}$ cut along these arcs is a disc (with corners), and since the mapping class group of a disc is trivial, a diffeomorphism of $\Sigma_{0,4}$ fixed on $\partial\Sigma_{0,4}$ is determined (up to an isotopy fixed on $\partial\Sigma_{0,4}$) by its action on the arcs I_1, I_2, I_3 . Thus one need only calculate the effect of both sides of (28) on I_i and see that the results agree up to isotopy fixed on $\partial\Sigma_{0,4}$. For simplicity, we only draw the pictures relevant for I_1 ; the reader can easily draw the corresponding pictures for the other two arcs.

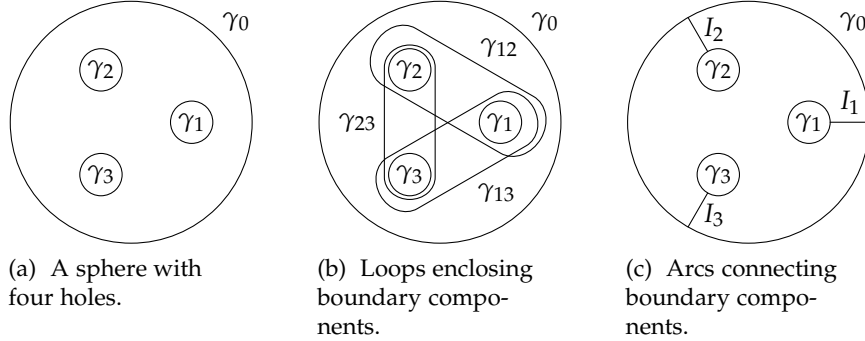


Figure 2: The lantern relation.

Now if we choose I_1 to be the horizontal line segment of Figure 2c, we see that τ_3 , τ_2 , and τ_{23} act trivially on I_1 . Thus we need only to show that $\tau_0 \tau_1$ and $\tau_{12} \tau_{13}$ has the same effect on I_1 . This is clear from the two rows of pictures in Figure 3 below. \square

Lemma A.5. Let $\Sigma_{1,2}$ be a two-holed torus, and let α, ε be non-intersecting simple closed curves which both intersect a simple closed curve β in a single point. Further, let δ, γ denote simple closed curves parallel to the boundary components (see Figure 4). Then

$$(\tau_\alpha \tau_\varepsilon \tau_\beta)^4 = \tau_\delta \tau_\gamma. \quad (29)$$

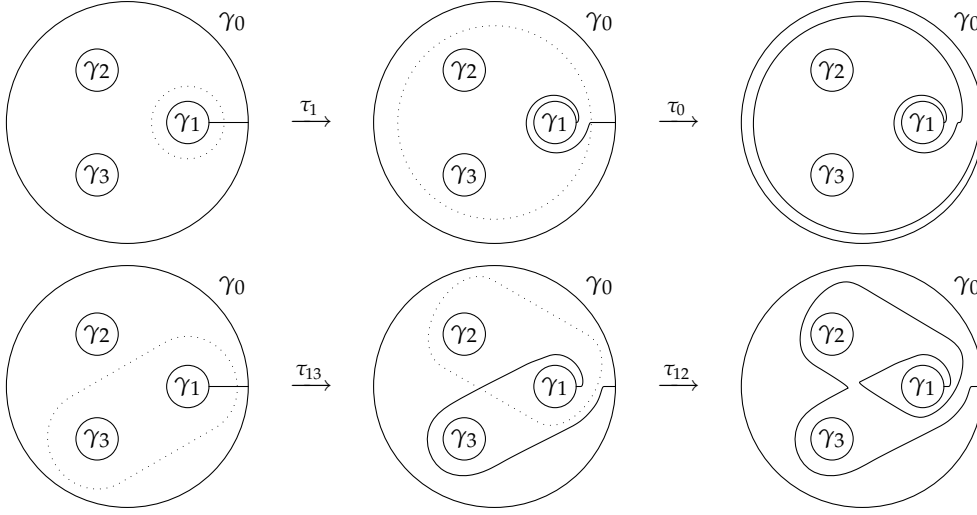
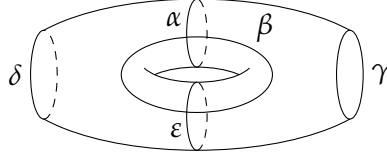
Figure 3: The effect of $\tau_0\tau_1$ and $\tau_{12}\tau_{13}$ on I_1 agree up to isotopy.

Figure 4: The two-holed torus relation

This lemma can be proved in a completely similar fashion as above by choosing a set of proper arcs such that cutting along these gives a disc (with corners), and proving that both sides of (29) has the same effect on this set of arcs.

These simple results have a few interesting consequences with respect to the homology of the mapping class group

Proposition A.6. *If $g \geq 2$, the Dehn twist on a boundary component of $\Sigma_{g,r}$ can be written in terms of Dehn twists on non-separating curves.*

Proof. The assumption on the genus implies that we may find an embedding of $\Sigma_{0,4} \rightarrow \Sigma_{g,r}$ such that γ_0 is mapped to the boundary component in question and the remaining six curves involved in the lantern relation are mapped to non-separating curves (think of $\Sigma_{g,r}$ as being obtained by gluing three boundary components of $\Sigma_{g-2,r+2}$ to γ_1 , γ_2 and γ_3 , respectively). Then the relation $\tau_0 = \tau_{12}\tau_{13}\tau_{23}\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}$ also holds in $\Gamma_{g,r}$. \square

Proposition A.7. *When $g \geq 2$, $\Gamma_{g,r}$ is generated by Dehn twists on non-separating curves.*

Proof. We already know that the mapping class group is generated by Dehn twists, so it suffices to show that a Dehn twist on a separating curve γ can be written in terms of twists on non-separating curves in Σ . If we assume $g \geq 3$, cut Σ along γ and apply Proposition A.6 to the component which has genus ≥ 2 .

Now, if $g = 2$ and γ separates Σ into a genus 0 and a genus 2 component, we may still apply Proposition A.6 to see that τ_γ is a product of twists on non-separating curves. Hence assume γ cuts Σ into two genus 1 components. If one of these components has a boundary component other than the one arising from the cut along γ , we may apply Lemma A.5 to see that τ_γ can be written in terms of twists on non-separating curves along with the twist on the additional boundary component. But the latter may be written in terms of twists on non-separating curves in the original surface. If there are no other boundary components, we may still apply Lemma A.5, since we can simply cut out a disc bounded by a trivial loop δ ; then the relation (29) degenerates to $(\tau_\alpha \tau_\varepsilon \tau_\beta)^4 = \tau_\gamma$. \square

Corollary A.8. *When $g \geq 3$, $H_1(\Gamma, \mathbb{Z}) = 0$, and when $g = 2$, $H_1(\Gamma, \mathbb{Z}) = \mathbb{Z}/10\mathbb{Z}$. In both cases, $H_1(\Gamma, \mathbb{Q}) = 0$.*

Proof. Assume $g \geq 2$. The first homology group with integral coefficients is the same as the abelianization of the group. By Proposition A.7, Γ is generated by Dehn twists on non-separating curves. Since any two non-separating curves are related by a diffeomorphism, such Dehn twists are always conjugate by Lemma A.2. Hence $H_1(\Gamma, \mathbb{Z})$ is cyclic, generated by any such Dehn twist τ . If $g \geq 3$, we may find an embedding of $\Sigma_{0,4}$ into Σ such that all seven curves occurring in the lantern relation (28) are non-separating, so the relation $4\tau = 3\tau$ holds in the abelianization, and $H_1(\Gamma, \mathbb{Z}) = 0$ when $g \geq 3$.

If $g = 2$, there is an embedding of the two-holed torus in Σ such that all five curves occurring in Lemma A.5 are non-separating. This implies that the relation $12\tau = 2\tau$ holds in $H_1(\Gamma, \mathbb{Z})$; or in other words that $H_1(\Gamma, \mathbb{Z})$ is cyclic of order dividing 10. To see that the order is in fact 10, we observe that there is an epimorphism $\Gamma_{2,r} \rightarrow \Gamma_2$ inducing an epimorphism of abelianizations $H_1(\Gamma_{2,r}, \mathbb{Z}) \rightarrow H_1(\Gamma_2, \mathbb{Z})$, so it suffices to prove the statement for a closed surface of genus 2. The easiest way to do this is to use one of the known presentations of the mapping class group; for instance the very symmetric presentation given by S. Gervais in [6]. \square

The first proof that $H_1(\Gamma, \mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z}$ is due to Mumford [9], and does not rely on a presentation of the mapping class group. We also remark that for $g = 1$ and $r \geq 1$, the result in Proposition 3.1(2) can also be obtained from the presentation given in [6].

Bibliography

- [1] Jørgen Ellegaard Andersen. Hitchin's connection, Toeplitz operators and symmetry invariant deformation quantization. 2006. arXiv:math.DG/0611126.
- [2] Jørgen Ellegaard Andersen, Josef Mattes, and Nicolai Reshetikhin. The Poisson structure on the moduli space of flat connections and chord diagrams. *Topology*, 35(4):1069–1083, 1996.
- [3] Jørgen Ellegaard Andersen, Josef Mattes, and Nicolai Reshetikhin. Quantization of the algebra of chord diagrams. *Math. Proc. Cambridge Philos. Soc.*, 124(3):451–467, 1998.
- [4] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.
- [5] Doug Bullock, Charles Frohman, and Joanna Kania-Bartoszyńska. Understanding the Kauffman bracket skein module. *J. Knot Theory Ramifications*, 8(3):265–277, 1999.
- [6] Sylvain Gervais. A finite presentation of the mapping class group of a punctured surface. *Topology*, 40(4):703–725, 2001.
- [7] John Harer. The third homology group of the moduli space of curves. *Duke Math. J.*, 63(1):25–55, 1991.
- [8] Nikolai V. Ivanov. Mapping class groups. In *Handbook of geometric topology*, pages 523–633. North-Holland, Amsterdam, 2002.
- [9] David Mumford. Abelian quotients of the Teichmüller modular group. *J. Analyse Math.*, 18:227–244, 1967.
- [10] Anders Reiter Skovborg. *The Moduli Space of Flat Connections on a Surface – Poisson Structures and Quantization*. PhD thesis, University of Aarhus, 2006.
Available from <http://www.imf.au.dk/publs?id=623>.